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On Mean Value Iterations with Application to Variational Inequality Problems

by
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Abstract

In this report, we show that in a Hilbert space, a mean value iterative process generated by a continuous quasi-nonexpansive mapping always converges to a fixed point of the mapping without any precondition. We then employ this result to obtain approximating solutions to the variational inequality and the generalized complementarity problems.

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1. Introduction

By using a technique of W. R. Mann [6], we show that in a Hilbert space, a mean value iterative process generated by a continuous quasi-nonexpansive mapping always converges to a fixed point of the mapping without any precondition. We then employ this result to obtain approximating solutions to the variational inequality and the generalized complementarity problems.

2. Preliminaries

Let K be a nonempty subset of a Hilbert space X. A mapping T from K into itself is said to be *quasi-nonexpansive* if T has a fixed point in K and for for any fixed point p of T in K, we have $||T(x) - p|| \leq ||x - p||$ for all $x \in K$. A nonexpansive mapping with a fixed point is clearly quasi-nonexpansive but not conversely. For example, let T be a mapping from $\mathbf{R}$ into itself defined by $T(0) = 0$ and $T(x) = x\sin(1/x)$ for $x \neq 0$. Then T is quasi-nonexpansive but not nonexpansive. Let $x_1 \in K$ be arbitrary. Given a sequence of real numbers $\{\alpha_n\}$ so that $0 \leq \alpha_n \leq 1$, and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, let $M(x_1, \alpha_n, T)$ be a sequence $\{x_n\}_{n=1}^{\infty}$ iteratively defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n).$$

Let $\mathbf{R}^n$ denote n-dimensional Euclidean space with the usual inner product and norm. Let $\mathbf{R}^n_+$ denote the nonnegative orthant of $\mathbf{R}^n$. Given a subset K of $\mathbf{R}^n$ and a mapping T from $\mathbf{R}^n$ into itself, the classical ***variational inequality problem***, denoted by $VIP(T,K)$ is to find a vector $x \in K$ such that

$$\langle u - x, T(x)\rangle \geq 0, \ \forall \ u \in K.$$

This original problem has been extensively studied in the past years. For details of the theories, algorithms and applications of the variational inequality problem, we refer readers to [3]. Basically, the task of the above problem is to find a vector $x \in K$ such that the image of x under the function f will form an acute angle with any vector with tail x and head $u \in K$.

Motivated by the work of Habetler and Price [2], Karamardian [5] introduced the following complementarity problem. Given a closed convex cone K of $\mathbf{R}^n$ and a mapping T from K into $\mathbf{R}^n$, the ***generalized complementarity problem***, denoted by $GCP(T,K)$, is to find a vector $x \in K$ such that

$$T(x) \in K^*, \ \langle x, T(x)\rangle = 0$$

where $K^* = \{y \in \mathbf{R}^n : \langle y, x\rangle \geq 0, \forall \ x \in K\}$ is the ***polar cone*** of K. When $K = \mathbf{R}^n_+$, $GCP(T,K)$ reduces to the following classical ***nonlinear complementarity problem*** which is denoted as $CP(T)$: find a vector $x \in \mathbf{R}^n$ such that

$$x \geq 0, \ T(x) \geq 0, \ \langle x, T(x)\rangle = 0.$$

The CP was first studied by Cottle [1] where the notion of ***positively bounded Jacobians*** was introduced and the proof was constructive in the sense that an algorithm was employed to compute the unique solution.

The variational inequality and the generalized complementarity problems are closely related to each other. In [5], Karamardian has shown that if the set K is a closed convex cone, then both the variational inequality problem $VI(T,K)$ and the generalized complementarity problem $GCP(T,K)$ have the same solution set.

3. The Main Results

We now state and prove the main results of this paper.

Theorem 3.1. *Let K be a nonempty compact and convex subset of a Hilbert space X and T be a continuous quasi-nonexpansive mapping from K into itself. Then for arbitrary $x_1 \in K$, the sequence $M(x_1, \alpha_n, T)$ converges to a fixed point of T.*

Proof. Let p be any fixed point of T. Without loss of generality, we may assume that $x_1 \neq p$. Since X is a Hilbert space, it can be shown that

$$||x_{n+1} - p||^2 = \alpha_n ||T(x_n) - p||^2 + (1 - \alpha_n)||x_n - p||^2 - \alpha_n(1 - \alpha_n)||T(x_n) - x_n||^2.$$

Since T is quasi-nonexpansive, $||T(x_n) - p|| \leq ||x_n - p||$ for all n. So

$$||x_{n+1} - p||^2 \leq ||x_n - p||^2 - \alpha_n(1 - \alpha_n)||T(x_n) - x_n||^2.$$

Hence for every positive integer m, $||x_{m+1} - p|| \leq ||x_m - p||$ and

$$||x_{m+1} - p||^2 \leq ||x_1 - p||^2 - \sum_{i=1}^{m} \alpha_i(1 - \alpha_i)||T(x_i) - x_i||^2$$

from which it follows that $\sum_{i=1}^{\infty} \alpha_i(1 - \alpha_i)||T(x_i) - x_i||^2 < \infty$. Hence

$$\lim_{n \longrightarrow \infty} ||T(x_n) - x_n|| = 0. \tag{1}$$

Since K is compact, the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$ with limit $q \in K$. From (1) it follows that q is a fixed point of T. For any $\epsilon > 0$, there exists an integer n_k so that $||x_{n_k} - q|| \leq \epsilon$. Then for all $n \geq n_k$, we have $||x_n - q|| \leq ||x_{n_k} - q|| \leq \epsilon$. Consequently, the entire sequence $\{x_n\}$ also converges to q, and the result follows.

In the case that the Hilbert space X in Theorem 3.1 is finite-dimensional, the compactness of the set K can be eliminated.

Theorem 3.2. *Let K be a nonempty closed and convex subset of a finite-dimensional Hilbert space X and T be a continuous quasi-nonexpansive mapping from K into itself. Then for arbitrary $x_1 \in K$, the sequence $M(x_1, \alpha_n, T)$ converges to a fixed point of T.*

Proof. Let p be any fixed point of T and let B be the closed ball of X with center p and radius $\|x_1 - p\|$. Then $x_n \in B$ for all n. Since B is compact, the sequence $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$ with limit $q \in B$. By the same argument as that in the proof of Theorem 3.1, it can be shown that q is a fixed point of T, and hence the result follows.

We note that the result of Theorem 3.2 may not be true if the Hilbert space X is infinite-dimensional. This is because that in this case, the set B in the proof of Theorem 3.2 is no longer compact (see, e.g., [4, Theorem 4, p. 180]).

Before we consider the application of Theorem 3.2 to the variational inequality problem, we need the following lemma. Let $K \subset \mathbf{R}^n$ be nonempty closed and convex. Let $P_K(\cdot)$ denote the orthogonal projection operator on K. That is, $P_K(x) = y$ where $\|y - x\| = \min_{v \in K} \|x - v\|$.

Lemma 3.3. *Let $K \subset \mathbf{R}^n$ be nonempty closed and convex and T be a mapping from K into $\mathbf{R}^n$.*

(i) *If $x \in K$ is a solution to $VI(T, K)$, then x is a fixed point of the mapping $P_{K,\rho}$ defined by $P_{K,\rho}(x) = P_K(x - \rho T(x))$, $x \in K$, for all $\rho > 0$.*

(ii) *If x is a fixed point of the mapping $P_{K,\rho}$ for some $\rho > 0$, then x is a solution to $VI(T, K)$.*

Proof. (i) If $x \in K$ solves $VI(T, K)$, then $\langle u - x, T(x)\rangle \geq 0$, $\forall\ u \in K$. Then for each $u \in K$ and for every $\rho > 0$, we have

$$\begin{aligned} \|u - x + \rho T(x)\|^2 &= \|u - x\|^2 + 2\langle u - x, \rho T(x)\rangle + \|\rho T(x)\|^2 \\ &\geq \|\rho T(x)\|^2. \end{aligned}$$

Therefore $x = P_{K,\rho}(x)$, and hence x is a fixed point of $P_{K,\rho}$.

(ii) Suppose that x is a fixed point of $P_{K,\rho}$ for some $\rho > 0$. Then

$$\|\rho T(x)\|^2 \leq \|u - x + \rho T(x)\|^2,\ \forall\ u \in K. \tag{2}$$

From (2), we have

$$\|u - x\|^2 + 2\rho\langle u - x, T(x)\rangle \geq 0,\ \forall\ u \in K. \tag{3}$$

Now for any $0 < \lambda < 1$ and any $u \in K$, we have $\lambda u + (1-\lambda)x \in K$. By substituting $\lambda u + (1-\lambda)x$ into (3), we have

$$\lambda^2 \|u - x\|^2 + 2\lambda\rho\langle u - x, T(x)\rangle \geq 0, \ \forall \ u \in K. \tag{4}$$

Finally by first dividing λ on both sides of (4) and then letting $\lambda \longrightarrow 0$, we have

$$\langle u - x, T(x)\rangle \geq 0, \ \forall \ u \in K.$$

Hence x is a solution to $VI(T, K)$.

Let T be a mapping from K into $\mathbf{R}^n$. T is said to be *β-monotone* on K if there exists an increasing function $\beta : [0, \infty) \longrightarrow [0, \infty)$ with $\beta(0) = 0$ and $\beta(r) \longrightarrow \infty$ as $r \longrightarrow \infty$ such that for any $x, y \in K$, we have

$$\langle T(x) - T(y), x - y\rangle \geq \|x - y\|\beta(\|x - y\|).$$

If $\beta(r) = kr$ for some $k > 0$, then T is said to be *strongly monotone* on K.

Theorem 3.4. *Let K be a nonempty closed convex subset of $\mathbf{R}^n$ and T be a continuous mapping from K into $\mathbf{R}^n$. Suppose that*

(i) *T is β-monotone on K,*

(ii) *there is some $\rho > 0$ so that $\rho\|T(x) - T(y)\|^2 \leq \|x - y\|\beta(\|x - y\|)$ for all $x, y \in K$.*

Then for arbitrary $x_1 \in K$, the sequence $M(x_1, \alpha_n, P_{K,2\rho})$ converges to the unique solution of $VI(T, K)$.

Proof. The fact that $VI(T, K)$ has a solution $p \in K$ follows from [8, Corollary 3.2.7]. To see that this solution is also unique, suppose that $q \in K$ is another solution of $VI(T, K)$. Then

$$\langle T(p), x - p\rangle \geq 0, \ \text{and} \ \langle T(q), x - q\rangle \geq 0, \ \forall \ x \in K. \tag{5}$$

From (5), we have

$$\langle T(p) - T(q), p - q\rangle \leq 0. \tag{6}$$

On the other hand, since T is β-monotone, there exist an increasing function $\beta : [0,\infty) \longrightarrow [0,\infty)$ with $\beta(0) = 0$ and $\beta(r) \longrightarrow \infty$ as $r \longrightarrow \infty$ such that

$$\langle T(p) - T(q), p - q\rangle \geq ||p - q||\beta(||p - q||). \tag{7}$$

From (6) and (7), we conclude that $p = q$.

Since the orthogonal projection operator is nonexpansive, by (i) and (ii) we have for all $x, y \in K$,

$$\begin{aligned} ||P_{K,2\rho}(x) - P_{K,2\rho}(y)||^2 &\leq ||(x - y) - 2\rho(T(x) - T(y))||^2 \\ &= ||x - y||^2 - 4\rho\langle T(x) - T(y), x - y\rangle + 4\rho^2||T(x) - T(y)||^2 \\ &\leq ||x - y||^2. \end{aligned}$$

Hence $P_{K,2\rho}$ is nonexpansive. By Lemma 3.3, p is the unique fixed point of $P_{K,2\rho}$. The result then follows from Theorem 3.1.

The following corollary immediate.

Corollary 3.5. *Let K be a nonempty closed convex subset of $\mathbf{R}^n$ and T be a continuous mapping from K into $\mathbf{R}^n$. If T is strongly monotone and Lipschitz continuous on K with constants k and L, respectively, then for arbitrary $x_1 \in K$, the sequence $M(x_1, \alpha_n, P_{K,\rho})$ converges to the unique solution of $VI(T, K)$, where $\rho = 2k/L^2$.*

Proof. By the assumptions, for all $x, y \in K$, we have

$$\langle T(x) - T(y), x - y\rangle \geq k||x - y||^2$$

and

$$||T(x) - T(y)|| \leq L||x - y||.$$

The result then follows from Theorem 3.2 by letting $\rho_1 = k/L^2$.

A version of Theorem 3.4 for the generalized complementarity problem is the following.

Theorem 3.6. *Let K be a closed convex cone of $\mathbf{R}^n$ and T be a continuous mapping from K into $\mathbf{R}^n$. Suppose that*

(i) *T is β-monotone on K,*

(ii) *there is some $\rho > 0$ so that $\rho\|T(x) - T(y)\|^2 \leq \|x - y\|\beta(\|x - y\|)$ for all $x, y \in K$.*

Then for arbitrary $x_1 \in K$, the sequence $M(x_1, \alpha_n, P_{K,2\rho})$ converges to the unique solution of $GCP(T, K)$.

Proof. This result follows from Theorem 3.4 and the fact that $VI(T, K)$ and $GCP(T, K)$ have the same solution set.

The following corollary is immediate.

Corollary 3.7. *Let T be a continuous mapping from $\mathbf{R}^n_+$ into $\mathbf{R}^n$. If T is strongly monotone and Lipschitz continuous on $\mathbf{R}^n_+$ with constants k and L, respectively, then for arbitrary $x_1 \geq 0$, the sequence $M(x_1, \alpha_n, P_{\mathbf{R}^n_+,\rho})$ converges to the unique solution of $CP(T)$, where $\rho = 2k/L^2$.*

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